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FOUR BRIEF EXAMPLES CONCERNING POLYNOMIALS ON CERTAIN BANACH SPACES

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Let E denote one of the spaces l^p $(1 \leq p < \infty)$ or c_0, and $\{e_1, e_2, \cdots\}$ be the standard basis in E. An element x in E will be written as $x = \Sigma_n x_n e_n$. The following examples are perhaps justified by the fact that their proofs are shorter than their statements.

Example A. *Suppose E is real,* and

$$\phi(t) = 3t^2 - 2t^3 \ (t \text{ real}), \qquad \phi_n(t) = \phi(\alpha_n t)/2^{n-1},$$

where $\alpha_n = 2^{n/4}$. *Then the mapping* $A(x) = \Sigma_n \phi_n(x_n)$ *is a continuous real-valued polynomial of degree* 3, *and the image of the critical points contains* $[0, 2]$.

Proof. Any x of the form $x = \Sigma_n \varepsilon_n \alpha_n^{-1} e_n$, where ε_n is 0 or 1, is a critical point of A, and $A(x) = \Sigma_n \varepsilon_n / 2^{n-1}$.

Example A is based on examples of Kupka [2] and Bonic [1], and the remark "bien sur" of Douady [Baton Rouge, April 1967].

Example B. *Suppose E is complex, and*

$$\begin{aligned} &\phi(z) = az^2 + bz^3 + cz^4 + dz^5, \\ &\phi_n(z) = \phi(\beta_n z)/2^{n-1}, \\ &\quad 4a = 4i + 4, \qquad 4b = -5i + 5, \\ &\quad 4c = -2i - 2, \qquad 4d = 3i - 3, \qquad \beta_n = 2^{n/6}, \end{aligned}$$

where z is complex. Then the mapping

$$B(x) = \Sigma_n \phi_n(x_{2n-1}) + \Sigma_n \phi_n(x_{2n})$$

is a continuous complex-valued polynomial of degree 5, *and the image of the critical points contains* $[0, 2] \times [0, 2]$.

Proof. Any x of the form

$$x = \Sigma_n \varepsilon_n \beta_n^{-1} e_{2n-1} - \Sigma_n \delta_n \beta_n^{-1} e_{2n},$$

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where ε_n and δ_n are 0 or 1, is a critical point of B, and $B(x) = \Sigma_n \varepsilon_n / 2^{n-1} + i\Sigma_n \delta_n / 2^{n-1}$.

Example C. *Suppose E is real or complex. Then the mapping $C(x) = \Sigma_n x_n^2 e_n$ is a continuous, but not completely continuous, polynomial of degree 2 from E into E, and each derivative of the polynomial is completely continuous.*

Proof. C is clearly continuous but cannot be completely continuous since $Ce_n = e_n$ for all n. Since $DC(x)h = \Sigma_n 2x_n h_n$ and $x_n \to 0$, we have that $DC(x)$ is completely continuous.

Example C answers a question asked the author by A. Tromba who pointed out that it solves negatively a problem posed in Vainberg [4, p. 51].

Example D. *Suppose E is real c_0. Then the mapping $D(x) = \Sigma_n (x_n + x_n^3) e_n$ is a continuous polynomial of degree three from c_0 into c_0, and is a proper map (the inverse image of a compact set is compact). Moreover, D has the form $D = I + D_0$, where D_0 is not completely continuous, but each derivative of D_0 is completely continuous.*

Proof. Letting ϕ denote the inverse of the mapping $\psi(t) = t + t^3$ we have that $D^{-1}(x) = \Sigma_n \phi(x_n) e_n$ is a continuous map of c_0 into c_0 and hence that D is proper. The facts about D_0 follow exactly as in Example C.

Example D is of some interest in degree theory since the usual Leray-Schauder degree is given for maps of the form $I + F$, where F is a completely continuous mapping. In [3] Tromba develops a degree theory for smooth proper maps of the form $I + G$ only assuming that each $DG(x)$ is completely continuous. Therefore this example gives an instance, where the latter but not the former degre is defined.

Bibliography

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